Tutorial 2 : Selected problems of Assignment 2

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Announcement

I will be on leave from 22 Sep to 30 Sep. During this period, please find me through my email ylli@math.cuhk.edu.hk if necessary.

Short lecture on Landau symbols

(reference: Wikipedia - Big O notation)

(1) Big-O notation

Let $f : \mathbb{R} \to \mathbb{C}$ be any function,

$g : \mathbb{R} \to \mathbb{R}$ such that $\exists K \in \mathbb{R}$ s.t.

$$g(x) > 0, \quad \forall |x| > K$$

Def We say that $f(x) = O(g(x))$ as $x \to +\infty$ (resp. $x \to -\infty$)

if $\exists M \in \mathbb{R}$, $\exists x_0 \in \mathbb{R}$ s.t.

$$|f(x)| \leq M \cdot g(x) \qquad \forall x \geq x_0, \text{ (resp. } x \leq x_0 \text{)}$$

Exercise: Equivalently, $\displaystyle \limsup_{x \to +\infty} \frac{|f(x)|}{g(x)} < +\infty$

$$\left(\text{resp: } \limsup_{x \to -\infty} \frac{|f(x)|}{g(x)} < +\infty \right)$$

Example: $f(x) = 2ix^4 - 3x + 4i$

then $f(x) = O(x^4)$ as $x \to \pm\infty$

(actually: $f(x) = O(|x|^n)$, $\forall n \geq 4$)

(2) small-o notation

Def We say that $f(x) = o(g(x))$ as $x \to +\infty$ (resp. $x \to -\infty$)

if $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t.

$$|f(x)| \leq \varepsilon \cdot g(x) \qquad \forall x \geq N, \text{ (resp. } x \leq N \text{)}$$

Exercise: Equivalently, $\displaystyle\lim_{x \to +\infty} \frac{|f(x)|}{g(x)} = 0$

$$\left(\text{resp: } \lim_{x \to -\infty} \frac{|f(x)|}{g(x)} = 0 \right)$$

Example: $f(x) = 2i x^4 - 3x + 4i$

then $f(x) = o(|x|^5)$ as $x \to \pm\infty$

but $f(x) \neq o(|x|^4)$

Remark There are variants of big-O and small-o notations in different settings. For instance, it makes sense to define

$(a_n) = O(b_n)$ or $(a_n) = o(b_n)$

where $(a_n)_{n \in \mathbb{Z}}$, $(b_n)_{n \in \mathbb{Z}}$ are bisequences.

(w/ $a_n \in \mathbb{C}$, $b_n \in \mathbb{R}$ s.t. $\exists K \in \mathbb{N}$ s.t. $b_n > 0$, $\forall |n| > K$)

Q1 (HW2, Q3)

Define the set of rapidly decreasing bisequences $\mathcal{C}^\infty$ by

(note: different from HW2)

$$\mathcal{C}^\infty = \{ (c_n)_{n\in\mathbb{Z}} \mid c_n \in \mathbb{C} ; \ \forall k \geq 0, \ (c_n) \in o(|n|^{-k}) \text{ as } n \to \pm\infty \}$$

$$\Updownarrow$$

$$\left(\lim_{n\to\pm\infty} \frac{|c_n|}{|n|^{-k}} = 0 \right)$$

Show that for any smooth 2π-periodic function f,

$$(\hat{f}(n)) \in \mathcal{C}^\infty$$

Sol: Fix $k \geq 0$, by Property 2 in Chapter 1, p.10, we have

$$\widehat{f^{(k+1)}}(n) = (in)^{k+1} \hat{f}(n)$$

$$\therefore \forall n \neq 0, \ |\hat{f}(n)| \leq \frac{|\widehat{f^{(k+1)}}(n)|}{|n|^{k+1}} = \frac{\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{(k+1)}(x) e^{-inx} dx \right|}{|n|^{k+1}}$$

$$\leq \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k+1)}(x)| dx}{|n|^{k+1}} = \frac{M_k}{|n|^{k+1}}$$

$$\therefore \lim_{n\to\pm\infty} \frac{|\hat{f}(n)|}{|n|^{-k}} \leq \lim_{n\to\pm\infty} \frac{M_k}{|n|} = 0$$

Hence $\forall k \geq 0$, $\displaystyle\lim_{n\to\pm\infty} \frac{|\hat{f}(n)|}{|n|^{-k}} = 0$. $\therefore (\hat{f}(n)) \in \mathcal{C}^\infty$

Q2 (HW2, Q6)

Define the set of L^2- bisequences $\mathcal{C}'$ by

$$\mathcal{C}' = \left\{ (c_n)_{n\in\mathbb{Z}} \;\middle|\; c_n \in \mathbb{C} \,;\, \sum_{n=-\infty}^{+\infty} |c_n|^2 < +\infty \right\}$$

$R_{2\pi} = \{\text{integrable } 2\pi\text{-periodic function}\}$ as vector space over $\mathbb{C}$.

Show that $\forall f \in R_{2\pi}$, Bessel Inequality holds:

$$\sum_{n=-\infty}^{+\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx \qquad \text{Hence, } (\hat{f}(n))_{n\in\mathbb{Z}} \in \mathcal{C}'.$$

Sol We first introduce an inner product space structure on $R_{2\pi}$

$$\langle\cdot,\cdot\rangle : R_{2\pi} \times R_{2\pi} \longrightarrow \mathbb{C} \quad \text{by} \quad \langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

Exercise: Show that $\langle\cdot,\cdot\rangle$ defines a Hermitian inner product on $R_{2\pi}$

and induced norm $\|\cdot\|^2 : R_{2\pi} \to \mathbb{R}$ by

$$\|f\|^2 := \langle f, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Then for each $N \in \mathbb{N}$, recall that $\{e^{inx}\}_{n=-N}^{N} \subseteq R_{2\pi}$ satisfies

$$\langle e^{inx}, e^{imx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}, \text{ i.e. } \{e^{inx}\}_{n=-N}^{N} \text{ is orthonormal}$$

Also, $\forall -N \le n \le N$, $\langle f, e^{inx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \hat{f}(n)$

Note that $0 \le \| f - \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \|^2$.

$= \langle f - \sum_{n=-N}^{N} \hat{f}(n) e^{inx}, f - \sum_{m=-N}^{N} \hat{f}(m) e^{imx} \rangle$

$= \|f\|^2 - \sum_{n=-N}^{N} \hat{f}(n) \langle e^{inx}, f \rangle - \sum_{m=-N}^{N} \overline{\hat{f}(m)} \langle f, e^{imx} \rangle + \sum_{-N \le m, n \le N} \hat{f}(n) \overline{\hat{f}(m)} \langle e^{inx}, e^{imx} \rangle$

$= \|f\|^2 - \sum_{n=-N}^{N} \hat{f}(n) \overline{\hat{f}(n)} - \sum_{m=-N}^{N} \overline{\hat{f}(m)} \hat{f}(m) + \sum_{l=-N}^{N} \hat{f}(l) \overline{\hat{f}(l)}$

$= \|f\|^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2$

$\therefore \forall N \in \mathbb{N}$, $\sum_{n=-N}^{N} |\hat{f}(n)|^2 \le \|f\|^2$,

$\therefore$ Take $N \to +\infty$, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le \|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$

Rmk (1) Bessel inequality holds true for any inner product space.

(2) In fact, equality holds true for any Hilbert space,

e.g. $(R_{2\pi}, \langle \cdot, \cdot \rangle)$ which is called Parseval Identity.